and

$$\frac{OB'}{PK} = \frac{ON}{PN};$$

whence

$$\frac{OB \cdot OB'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{1}$$

Also from the similar triangles OAL and PKL, combined with the similar triangles ONA' and MPN, we have

$$\frac{OA \cdot OA'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{2}$$

By combining (1) and (2) we have the fundamental relation

$$OA \cdot OA' = OB \cdot OB',$$

or in words:

The product of the distances from the center to a pair of conjugate points in the involution is constant.

From this theorem all of the usual developments may easily be made. The corresponding theory of lines in involution may be obtained in the same way, or by applying the principal of duality.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

The Pell Equation. By EDWARD EVERETT WHITFORD, Instructor in Mathematics in the College of the City of New York. Whitford, New York, 1912. iv + 193 pages. $1.00 postpaid.

Let A be a given non-square positive integer and consider the equation

$$x^2 - Ay^2 = 1 \tag{1}$$

from the point of view of determining the positive integers x, y which satisfy the equation. This is the so-called Pellian problem. Equation (1) is referred to as the Pell equation.

The principal purpose of the book under review is to give a history of the Pell equation (1) and of the more general equation $x^2 - Ay^2 = B$, where B is a given integer (positive or negative). There is added a table of the simplest solution of (1) for each value of A from 1,501 to 1,700 inclusive, similar tables having previously been given by other authors for the values of A less than 1,501. The book contains also a bibliography of the Pell equation, with references to over 300 authors, and a table of continued fractions for $\sqrt{A}$.

Of the historical section, covering pages 1–101 of the text, we shall speak further.

The author begins with a discussion of Euler's error by which the name of Pell improperly came to be associated with equation (1). There follows an interesting discussion of the relation of the Pellian problem to that of approximating the square root of a non-square integer. An account is then given of the most ancient Hindu solutions (pp. 6–9), of the most ancient Greek solutions (pp. 9–13), of the work of Theon (pp. 13–15), Archimedes (pp. 15–21), Heron (pp. 21–22) and Diophantus (pp. 22–26). Emphasis is put upon the later work of the Hindus (pp. 26–39), and brief mention is made of that of the Arabs (pp. 39–41) and of the Europeans before the time of Fermat (pp. 41–46).

This history of the early and largely unsuccessful struggle of mathematicians with the difficulties of the Pellian problem is very instructive for those who are engaged now in the development and extension of science and knowledge.

The French mathematician Fermat (1601–1665) was the first to assert that equation (1), where A is any non-square integer, always has an unlimited number of solutions in integers. To prove this and to obtain a method for finding all of these solutions he proposed it as a challenge problem to the English mathematicians. From his remarks about it, it is clear that he considered it a very difficult problem.

This challenge problem of Fermat's was solved by Lord Brouncker, at least so far as giving means for finding the solutions is concerned. On account of this success, the English mathematician Wallace congratulates his fellow-countryman Brouncker that he has "preserved untarnished the fame which Englishmen have won in former times with Frenchmen and has shown that England's champions in wisdom are just as strong as those in war." Fermat himself never published his solution of the problem.

It was Euler who first recognized the deep importance of the Pell equation for the general solution of the indeterminate equation of the second degree. He left several memoirs dealing with this question.

It was Lagrange who first proved in a rigorous manner that (1) always has solutions in integers. Of the work of Lagrange, Legendre said that it "must be considered the most important step which has been made up to the present time in the indeterminate analysis."

Our author treats further the work of Gauss and Dirichlet and of the relation of the Pellian problem to several other disciplines, namely, the following: quadratic forms, circle division, elliptic functions and hyperbolic functions.

The book on the whole will repay perusal both by the working mathematician and by the amateur in the theory of numbers.

R. D. CARMICHAEL.